REVIEWS

Collected works of Theodore von Kármán. Vol. I, 1902–13, pp. 530; vol. II, 1914–32, pp. 436; vol. III, 1933–39, pp. 391; vol. IV, 1940–51, pp. 480. London: Butterworths Scientific Publications, 1956. £14. 10*s*. the set.

These four massive volumes contain the one hundred and eleven published papers and lectures of Dr von Kármán during the fifty years 1902–51. While they are an astonishing tribute to the learning, insight, skill and energy of the author, they form only an interim report on his scientific activities, which are still continuing with unabated vigour and enthusiasm.

To review these volumes is as if one attempted to review the *Encyclopaedia Britannica* or passed judgement on the collected works of Leonardo da Vinci. To attempt any critical comment seems impertinent presumption; to relapse into a few phrases of appreciation and gratitude seems miserably inadequate.

Perhaps the most useful task the reviewer can attempt is to provide classified summaries of von Kármán's work, so that the research worker can easily locate the references he may be seeking and quickly envisage the rapid developments in von Kármán's thought. The references in the following brief notes on this great work give the date of publication and the number of the paper in the newly printed volumes.

Compressible flow

The fundamental parameter which discriminates between the different regimes of fluid flow is the Reynolds number $R = Ul/\nu$. The physical significance of this number was pointed out by von Kármán in 1923 (**38**). According to the kinetic theory of gases the kinematic viscosity is given approximately by the formula $\nu = c\lambda$, where c is the mean molecular velocity and λ is the mean free path. Hence

$$R = (U/c)(l/\lambda).$$

Now c is of the same order as the acoustic speed, so that U/c is of the same order as the Mach number M; while l/λ measures the size l of a typical dimension of the moving body in terms of the molecular scale. We can distinguish three regimes:

(*a*) the region of Brownian movement for which $U/c \ll 1$, $l/\lambda \sim 1$;
(*b*) the region of hydrodynamic flow for which $U/c \ll 1$, $l/\lambda \gg 1$;
(*c*) the ballistic region for which $U/c \sim 1$, $l/\lambda \gg 1$.

Curiously enough, one of von Kármán's earliest papers (1907, **3**) deals with the 'ballistic regime', and discusses stationary waves in streams of gas in terms of the wave equation

$$(1-M^2)f_{xx}+f_{yy}+f_{zz} = 0,$$

where f (a little surprisingly) is not the velocity potential (the existence of which is not assumed), but the logarithm of the density ρ.

His next paper (1908, **4**) discusses the flow of gas and steam flow through a de Laval nozzle, the formation of shock waves and the Schlieren method for the optical examination of the flow. The diagrams (figures 9 and 10) in this paper must surely be some of the earliest diagrams constructed by aerodynamicists to illustrate the differences between the propagation of acoustic waves in subsonic ($M < 1$) and supersonic conditions ($M > 1$).

In 1932 (**56**) von Kármán published a purely mathematical paper with Norton B. Moore on the wave resistance of supersonic bodies of revolution. This paper has had a great influence on aerodynamic theory, especially as regards the analogy between the von Kármán–Moore formula for wave drag and the classical formula for the induced drag of a lifting line (in **95**). A paper written in 1935 (**67**) gives a fine account of the general theory of the problem of resistance in compressible fluids, including both boundary layer effects and heat transfer.

von Kármán returned to the subject of compressibility effects in gas dynamics in 1941 (**84**), when he gave a general survey of the theory and of the analytical methods of solution (expansion in powers of M, Glauert's transformation, the hodograph method, the Tsien approximation (suggested by von Kármán in 1939)) and discussed the variation of drag and lift in the transonic region and the existence of a critical Mach number. The same themes are developed in a general survey given at the University of Pennsylvania Bicentennial Conference (no date, **85**), and with special reference to supersonic flow in 1947 (**95**). This long and important paper presents von Kármán's 'signal theory', an account of the mechanism of lift and wave drag, a summary of wave-drag calculations, the linearized theory, boundary-layer effects, transonic flow and interaction between boundary layers and shock waves.

The analogy between supersonic gas flow and the 'shooting' or supra-undal flow of water in an open channel was developed and exploited by von Kármán (1938, **72**) in a paper which emphasizes the practical applications of this fruitful comparison, which had been described by Preiswerk (at the suggestion of Ackeret) in the previous year.

von Karman wrote only two papers on transonic flow ($M \sim 1$), but one of these (1947, **96**) has been of outstanding importance as it gave for the first time the similarity law which replaces, for transonic flow, the Glauert rule for incompressible flow and the Ackeret rule for supersonic flow. The other paper (1950, **107**) gives a special, simple and exact solution of the Tricomi equations for transonic flow along an undulating wall.

A region of flow not mentioned in von Kármán's 1923 paper (**38**) is that of free molecule flow ($l \sim \lambda$), but in fact von Kármán had already entered this region in 1913 (**22**), in a paper written in collaboration with H. Bolze and M. Born on the temperature jump at the surface of a body exposed to molecular bombardment.

Incompressible inviscid flow

Although the theory of incompressible flow was not von Kármán's main interest, his contributions to this subject are still of considerable significance. His earliest paper (1918, **29**), written in collaboration with E. Trefftz, generalizes the Joukowski transformation of a circular cylinder into aerofoil sections. The von Kármán–Trefftz transformation

$$(z-A)/(z-B) = [(\zeta-A)/(\zeta-B)]^k$$

reduces to the Joukowski transformation when $k = 2$, and, when k is slightly less than 2, gives a profile with a trailing edge angle of $\delta = 2\pi - k\pi$.

The next paper (1927, **45**) gives a method of calculating the pressure distribution around airship bodies. A piece-wise constant source distribution is assumed along the axis of symmetry and the source strengths are adjusted to give the correct profile at a suitable number of stations. Both longitudinal and transverse motions are considered. Further developments of this work and practical applications are summarized in von Kármán's opening publication from the Daniel Guggenheim Airship Institute (1933, **59**).

A short note (1929, **48**) discusses the effect on the lift of an aerofoil of a discontinuity in the velocity of the wind; and a later note (1942, **86**) gives some comments on Newton's theory of air resistance.

A new presentation of lifting line theory given in 1935 (**66**) starts from the Fourier transform of the potential function and gives an extension to the theory of the lifting surface.

A similar technique is employed in a paper written with H. S. Tsien (1945, **92**), which extends Prandtl's lifting line theory to a wing in a non-uniform flow such as occurs in an open jet wind-tunnel or in a propeller slip stream.

In 1938 (**74**) von Kármán, in collaboration with W. R. Sears, made a most illuminating contribution to the theory of the non-uniform motion of an aerofoil, initiated by H. Wagner in 1925. von Kármán's paper treats specifically the case of a thin aerofoil accelerated from rest, and calculates the lift and pitching moment directly from the formula for the linear and angular momenta of the vortex system. Applications to oscillatory motion and to sharp-edged gusts complete a paper which is notable for its continuous emphasis on the physics of the motion and its remarkable simplifications of the mathematical theory.

The last paper von Kármán wrote on classical hydrodynamics (1949, **104**) gives a special solution of the problem of the closed wake behind a flat plate accelerated along its normal.

In this section we must also include a pioneer paper on the impact of seaplane floats during landing (1929, **50**), written in collaboration with Frank L. Wattendorf. This paper makes a bold application of the concepts of virtual mass and of pressure propagation in a compressible fluid, and has stimulated much research.

Three famous papers (1911, **15**; 1912, **16**; 1912, **17** (with H. Rubach)) must be classified under the heading of incompressible, inviscid flow, although their topic is the mechanism of the production of drag; for they deal with the von Kármán 'vortex street' in a perfect fluid. The pioneer investigation of the first of these papers is superseded and corrected by the results of the second and third. The theory is now well known and provides a convincing picture of the relation between the vortices shed periodically into the wake from a bluff body and the drag experienced by the body in a certain range of the Reynolds numbers. Even now we have no means of completing von Kármán's theory by calculating the frequency with which the vortices are shed.

Aviation

von Kármán's interest in aerodynamics was far from purely theoretical, and he took great interest in the technical problems of aviation and of aeronautical engineering. One of his earliest papers (1908, **5**) describes the advantages of the very 'light-weight' engines which were being manufactured in Paris. The next paper on aeronautics (1914/15, **27**), written in collaboration with E. Trefftz, gives a careful and systematic account of the longitudinal stability and oscillations of an aircraft.

von Kármán was acquainted with the pioneer investigations of F. W. Lanchester and devoted a paper (1921, **33**) to the analytical theory of the 'switchback model' devised by the latter to represent the dynamic soaring of birds (*Aerodonetics*, London, 1908, p. 277).

von Kármán's practical activities in the field of helicopter design during the years 1916–17, while he was in the Austro-Hungarian Army Air Force, issued later in a long theoretical paper (1921, **34**) on various problems of helicopter flight. A few years later (1924, **41**) he wrote, with Th. Bienen, a long and detailed study of propeller theory.

His other main contributions to research in aviation are a lengthy paper, written with Clark B. Millikan (1934, **62**), on the use of the wind-tunnel in connexion with problems of aircraft design, and an aeronautical conference paper (1947, **97**) on the theory of stability and control at high speeds. The four last-mentioned papers contain so much detailed information that it is difficult to give more adequate summaries.

With his unique experience of the aeronautical world, his profound physical insight and engineering ability, von Kármán was admirably equipped to give magisterial exposition of the development of aviation, and he splendidly fulfilled this task in a number of addresses (1931, **54**; 1932, **58**; 1943, **88**; 1948, **103**).

Fuels and combustion

von Kármán's present interest in problems of fuel and combustion and his recent contributions to this subject are so actual that it is just a little surprising to find him writing a paper with Frank J. Malina in 1940 (**82**) on the characteristics of the ideal solid propellent rocket motor. In 1943 (**89**) and 1944 (**90**) he was active in guiding the preparation of memoranda on long-range rocket projectiles and on jet-propulsion systems. A most remarkably prophetic article published in 1945 (**93**) excitedly foresees the application of nuclear energy to propulsion. An interesting lecture delivered in 1950 (**105**) with G. Gabrielli gives a comparative study of the specific power required for the propulsion of various types of land, sea and air vehicles. Finally, the last paper included in these volumes (1951, **111**), written with G. Millán, lays the foundations of the thermal theory of constant pressure deflagration.

Turbulent and laminar flow

von Kármán's contributions to the theory of turbulent and of laminar boundary layers have exercised a profound influence on the development of this difficult and important branch of fluid dynamics. Broadly speaking they fall into four sections in chronological order:

(1) the search for 'universal' laws of turbulence, phenomenological in character, and (in fact) rather limited in range;

(2) the construction and use of the 'momentum equation' for boundary layer flow;

(3) 'mixing length' theories, stemming from Prandtl's investigations; and

(4) the theory of 'inner' and 'outer' solutions of the laminar boundary layer equations.

The first paper (1911, **14**) suggests an empirical relation between the coefficient of turbulent viscosity and the coefficient of laminar viscosity in which various fluids are compared with water. The relation has the form

$$(\mu_S/\mu_W)_T = (\mu_S/\mu_W)_L^{2n-1}(\rho_S/\rho_W)^{1-n},$$

where $n = 0{\cdot}6173$, the suffixes S and W refer to the fluid considered and water, while T and L refer to turbulent and laminar conditions.

The famous momentum equation, which has dominated boundary layer theory for so many years, first appears in 1921 (**32**), together with a dimensional argument leading to the 'one-seventh power law' for the velocity distribution in the turbulent boundary layer of cylindrical pipes. The combination of these two concepts to turbulent flow over a flat plate gave a friction coefficient c_f proportional to $R^{-\frac{1}{5}}$, R being the Reynolds number based on the plate length. The same paper also applies the method of the momentum integral to the laminar and turbulent boundary layers on a rotating plate. Similar ideas are developed in an expository article (1922, **36**).

von Kármán's first attempt to construct a statistical theory of turbulence was made in 1924 (**42**) in which the 'probability' $f(\alpha,\beta)$ of the motion described by the stream function $\psi = A\cos(\alpha x+\beta y)$ was estimated by maximizing the integral $\int\int f\log f\,d\alpha\,d\beta$, subject to a condition of energy balance.

'Mixing length' theories first appear in 1928 (**46**) in connexion with drag experiments on long bodies, and a systematic account is given in 1930 (**52** and **53**), together with applications of von Kármán's relation $l \sim U_0'/U_0''$. Further developments are given in (1932, **57**), especially with regard to the 'logarithmic turbulent velocity profile' and the laminar sublayer near the solid boundary; and in (1934, **61**) where the validity of a universal velocity distribution is closely examined, with especial reference to the effect of roughness. A long and valuable historical summary is given in (1934, **64**).

In 1934 (**63**) von Kármán, in collaboration with Clark B. Millikan, developed a new method of obtaining approximate solutions of the laminar boundary layer equations, in which a modification of von Mises's equation was used to obtain a criterion for laminar separation. The inner and outer regions of the boundary layer were treated separately and joined at the inflexion point of the velocity profile. Applications were made to certain 'roof top profiles', and in a subsequent paper by the same authors (1935, **65**) to the problem of the maximum lift of certain airfoils.

The effects of compressibility on boundary-layer theory are studied in 1935 (**67**) and in a paper (1938, **73**) published in collaboration with H. S. Tsien. Finally, in 1939 (**76**) von Kármán gave an important extension of Reynolds's analogy between momentum and heat transfer to the case of a liquid flowing over a solid wall.

The last group of papers in this section deals with statistical theories of turbulence. The first of these (1937, **68**) develops Sir Geoffrey Taylor's early theory (1935) of the correlation function $R(y) = \overline{u_1 u_2}/\overline{u^2} = 1 - \lambda^{-2} y^2 + \ldots$ by obtaining a second relation between $\overline{u^2}$ and $\overline{\lambda^2}$. The next paper (1937, **69**) extends Taylor's general theory of isotropic turbulence by introducing the correlation tensor $R_{ik} = \overline{u_i u_k'}/\overline{u^2}$, discussing the decay of turbulence, and giving a new theory of turbulent shear motion. The Wilbur Wright Memorial Lecture (1937, **70**) gives a splendid account of turbulence and its relevance for the aeronautical engineer. The general mathematical theory is given in a joint paper (1938, **71**) written with L. Howarth, which provides the basic theory of the correlation tensor for isotropic turbulence, and the general theory of the decay of turbulence. Further remarks on this theory are given in (1938, **75**).

Some ten years later (1948, **99**, **101**), von Kármán returned to this subject using the powerful methods of Fourier analysis (Taylor, 1938) for the study of the correlation function and the velocity distribution. Together with C. C. Lin (1951, **108**) he wrote a survey of recent developments in the statistical theory and proposed a simple physical picture of the process of the decay of turbulence. The last paper on turbulence in these volumes is paradoxically entitled 'Introductory Remarks...' (1951, **109**), a title which in fact is most appropriate to the introductory paper at a symposium on cosmical aerodynamics.

Elasticity and strength of materials

Perhaps the greatest volume of von Kármán's research is that devoted to theoretical and experimental elasticity and to problems relating to the strength of materials.

Some of the earliest of these works are articles written for the *Encyklopädie der mathematischen Wissenschaften* on problems of stability in machine construction (1910, **10**), on the physical basis of the theory of stability (1910, **11**), in collaboration with L. Föppl, or for the *Handwörterbuch der Naturwissenschaften* on Elasticity (1913, **23**), Stability (1913, **24**), Equilibrium (1913, **25**), and Hardness (1913, **26**). The first of these articles (**10**) is remarkable for giving (for the first time) the now famous 'von Kármán non-linear equations' for the transverse deflexion of a thin plate loaded by edge thrusts and moments together with transverse loads. These equations appear in vol. I, p. 177 (equation 29).

An early paper (1909, **7**) (written with Alfred Haar) gives a general theory of stress distribution in plastic and granular material and derives the basic equations from a minimum energy principle supplemented by an inequality expressing the conditions for the elastic or plastic regime. A short note (1916, **28**) is devoted to hysteresis in materials. The theory of elastic 'grenzzustände' is taken up again in 1926 (**44**) with special reference to earth pressure. A comparatively recent paper (1950, **106**) discusses in some detail the propagation of plastic deformation in solids.

von Kármán wrote only one paper on technical processes, viz. a study of the theory of rolling processes (1925, **43**).

However, von Kármán's major contribution to the study of elasticity is undoubtedly the magnificent series of papers on problems of buckling and stability, especially of thin plates and tubes.

The earliest of these investigations (1906, **2**, and 1908, **6**) dealt with the classical problem of the buckling of a straight strut and showed that the discrepancy between the Euler theory and experiment disappears if one employs the 'tangent' modulus of elasticity. After a short paper (1909, **8**) on the strength of corrugated fire tubes, von Kármán returned to this topic in his Inaugural Dissertation at Göttingen (1910, **9**) in which he gave a general survey of his research on the buckling of struts. von Kármán's experimental work on the failure of materials under triaxial stress is given in (1911, **12**) and a further study of thin-walled tubes appeared in the same year (**13**).

The theory of deep thin beams was investigated by von Kármán (1923, **37**) using a two-dimensional theory and the theory of T-section beams in 1924 (**40**). A joint paper with K. Friedrichs (1929, **49**) discusses the stressing of aircraft wings.

The next pair of papers is concerned with the post-buckling behaviour of thin metal sheets and shells—a subject of great importance for the aeronautical engineer. Together with Ernest E. Sechler and L. H. Donnell, von Kármán investigated the strength of thin plates under compression assuming the load to be carried by two strips along the edges parallel to the external thrust (1932, **55**). A contribution to a symposium on problems of thin-walled structures (1933, **60**) illustrates some applications of Wagner's tension field theory.

In a paper written with H. S. Tsien (1939, **77**), von Kármán broke new ground by giving a theory of the failure of spherical shells due to 'oil-canning' or 'Durchschlag', i.e. a sudden dimpling of the shell with finite deformation. This topic is considered again in a paper written with Louis G. Dunn and Hsue-Shen Tsien (1940, **80**) which also gives numerous experimental results. Similar ideas were used to account for the buckling of thin cylindrical shells under axial compression in a paper also written with Tsien (1941, **83**).

Finally we must note a few papers on mathematical methods appropriate to problems of elasticity: a brief note of Ritz's method (1953, **21**), a useful survey of the use of series of orthogonal functions in structural problems (1939, **78**), and two elaborate papers (1944, **91** and 1946, **94**), written with N. B. Christensen and Wei-Zang Chien, respectively, on practical methods of computation for the analysis of thin-walled sections with variable twist when loaded in torsion.

Theory of crystals

von Kármán wrote three papers in collaboration with Max Born on the theory of molecular vibrations in crystals. In the first paper (1912, **18**) the theory is applied to optical properties and to the specific heat of crystals with cubic lattices. The next paper (1913, **19**) gives a more elaborate theory of specific heat; and the third paper (1914, **20**) examines the distribution of the characteristic vibrations of a crystal lattice.

'Philosophy' of engineering

von Kármán's general philosophy of engineering as a science and an art can be gathered from a group of some six papers which begin with a biographical notice of Joseph Popper ('Lynkeus') (1918, **30**), and a short article on the significance of Mechanics for the study of the technical physicist (1921, **31**). To this subject he returned again in a lecture given in 1930 (**51**) on Mathematics and Technical Science.

An illuminating and sympathetic comparison of the intellectual approaches to a problem made by engineers and mathematicians is given in a lecture (1940, **79**) and in the first paper published by the Quarterly of Applied Mathematics (1943, **87**).

The Willard Gibbs Lecture (1940, **81**) gives a lengthy, penetrating and stimulating account of the non-linear problems encountered by the engineer in vibration theory, problems of buckling, plasticity, water waves, jets and cavities, viscous boundary layers, and compressible flow: and clearly illustrates the effective use of analytical methods.

Miscellaneous investigations

A number of interesting papers do not fall conveniently under any of the preceding headings and, with some diffidence, are grouped together here. In this group we must include von Kármán's first paper (1902, **1**) on the motion of a heavy rod supported on a hemispherical end by a rough horizontal plate; a paper on thermo-electric equilibrium in solid insulators (1924, **39**); some remarks on the formation of sand ripples (1947, **98**); and a boundary layer theory of the damping of the oscillations of a viscous fluid in a U-tube (1948, **100**).

There is also an important paper (1948, **102**) on a number of problems of aerodynamic engineering, including the air resistance of vehicles, and the stressing of buildings and bridges for wind pressure and vortex shedding.

Certain characteristic features stand out clearly in this rapid survey of von Kármán's scientific work. It is impossible not to be impressed by the extraordinary prescience which he showed in recognizing at once the importance of all the major developments in fluid mechanics and elasticity, even in their earliest stages. Few of his papers 'date'. They all still possess an actuality and freshness which invigorates the reader. It is also characteristic of von Kármán's researches that each one seems to spring naturally and inevitably from some problem of practical engineering. Many of the papers were written in collaboration with younger men, all of whom would readily recognize the good fortune which thus brought them under von Kármán's benign influence.

Probably the best thanks we can give to him for publishing these researches is to point out that they bring the tale only up to 1951 and that it is high time that we had a fifth volume.

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